Entropy Bound for the TM Electromagnetic Field in the Half Einstein Universe

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Abstract

An explicit calculation is given of the entropy/energy ratio for the TM modes of the electromagnetic field in the half Einstein universe. This geometry provides a mathematically convenient and physically instructive example of how the electromagnetic and thermodynamic quantities behave as a function of the nondimensional parameter $\delta = 1/2\pi a T$, a being the scale factor and T the temperature. On physical grounds (related to the relaxation time), it is the case of small δ's that is pertinent to thermodynamics. We find that as long as δ is small, the entropy/energy ratio behaves in the same way as for the TE modes. The entropy is thus bounded. The present kind of formalism makes it convenient to study also the influence from frequency dispersion. We discuss an example where a sharp cutoff dispersion relation can in principle truncate the electromagnetic oscillations in the Einstein cavity such that only the lowest mode survives.

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I. INTRODUCTION

Valuable insight can often be obtained by analysing models that are mathematically simple but nevertheless able to show the essentials of the physical properties of a system. The purpose of the present paper is to present an explicit calculation of the energy E^{TM} associated with the transverse modes (TM) of the electromagnetic field in a spherically symmetric volume endowed with the metric of the half Einstein universe, and to combine this with the analogous transverse electric mode result E^{TE} calculated earlier [1], in order to obtain the total electromagnetic energy $E = E^{TM} + E^{TE}$ in the volume.

The calculations will be given for a finite temperature T. One of our motivations is to show, in the case of the TM modes as well as in the previous case of TE modes [1], that the ratio of entropy S^{TM} to thermal energy E^{TM} is limited in the following sense:

$$\frac{S^{TM}}{E^{TM}} = \frac{4}{3}\beta, \tag{1}$$

at high temperatures. Here $\beta = 1/T$. (Geometric units are used.) The half Einstein universe, having received considerable attention in the past (see, for instance, Refs. [1, 2, 3, 4]), is both mathematically convenient to handle and provides an instructive physical example of how the entropy content is limited. The situation to be considered here thus falls into line with the current discussion in general about how entropy and energy behave both quantum mechanically and thermally in conformal field theories [5, 6, 7]. In particular, a topic that has attracted particular interest is the Verlinde bound for the ratio of entropy to energy [5]. To what extent the entropy bound is realized in physical situations is usually not clear beforehand; it has to be investigated by concrete calculations in each case. Previous calculations on the entropy bound have usually been done in the high temperature approximation [6, 7]. We shall also be assuming that T is high. This is actually a constraint that follows from thermodynamics: in order for thermal equilibrium to be established, T has to be much higher than the inverse transit time for light across the linear dimension of the system. This point is discussed in more detail in section II.

As mentioned, the case of transverse electric (TE) modes in the half Einstein universe was considered in Ref. [1], and the bound of Eq. (1) was verified in that case. The analogous result in the TM case is thus not unexpected, although the combined electromagnetic and thermodynamic behavior is complicated and can hardly be understood merely by inspection, without calculation. As we shall see, at higher orders in δ (see Eq. (3) below), the TM and

TE modes behave somewhat surprisingly, in the sense that the terms involving T^3 and T^2 in the energy do not cancel each other. This contrasts the behaviour known from Casimir theory in flat space, where the TE and TM modes compensate each other with respect to the mentioned terms and only the term proportional to T survives in E to the leading order (cf., for instance, equation (8.39) in [8] or equation (4.44) in [9]). The reason for this behaviour has to be related to the geometry of the half Einstein universe.

An earlier version of the present paper [10] actually gave an impetus to the recent investigation of Dowker on general spacetimes [11]. Dowker's analysis is much more general than the concrete example on Maxwell fields considered in the present paper. We think, however, that there are definite advantages of going into concrete detail as we are doing here: for one thing, it becomes easy to analyse the influence from *dispersion* in the wall material. Usually in field theory, one assumes that the walls have infinity conductivity for all frequencies. This assumption is unphysical, as there is always a frequency dispersion present. Of importance in a Casimir context, is the behaviour of the permittivity $\varepsilon(i\zeta)$ as function of positive imaginary frequency ζ. For a dielectric, the susceptibility $\varepsilon(i\zeta)-1$ is in essence proportional to $(1+\zeta^2/\omega_0^2)^{-1}$, where ω_0 is the (dominant) resonance frequency. Typically, $\omega_0 \approx 1.5\times 10^{16}$ rad/s. Thus ω_0 acts as a soft high-energy cutoff. When the frequencies are much higher than the resonance frequency one has $\varepsilon(i\zeta)-1=\omega_p^2/\zeta^2$ for all bodies, metals or dielectrics, where ω_p is the plasma frequency. Taking gold as an example, $\omega_p = 1.37\times 10^{16}$ rad/s. For very high frequencies the photons do not "see" the metal, and the number of modes in the cavity is therefore truncated. We shall return to this point in the final section.

Let us also make some remarks on conformal invariance. The electromagnetic field is known to be conformally invariant in $D=4$ spacetime dimensions. Thus the trace of the electromagnetic energy-momentum tensor is zero when D=4. In higher dimensions the situation becomes more complicated. The Casimir energy for $D>4$ was first calculated by Ambjørn and Wolfram [12]. We refer also to the recent analysis of Alnes *et al.* on the electromagnetic field between two parallel hyperplanes in higher dimensions, considering both metallic and MIT boundary conditions [13]. Using the axial gauge, the pressure between the plates was found to be constant, while the energy density was found to diverge at the boundaries. This peculiar behaviour is a direct consequence of the lack of conformal invariance when $D>4$.

Another related point worth noticing is the growing interest in higher derivative elec-

trodynamics, such as the Lagrangian form $(F + 1/F)$ studied by Novello *et al.* [14] (here $F \equiv F_{\mu\nu}F^{\mu\nu}$). One of the motivations of this kind of approach, among other things, is to model a phase of cosmic current acceleration. It would be of interest to apply such a theory also to the half Einstein universe, although we will not go further with this topic here. The approach has strong similarities with the $(R + 1/R)$ theory in gravity, which has been thoroughly studied in the recent past (see, for instance, papers of Nojiri and Odintsov [15]).

II. THE HALF EINSTEIN UNIVERSE

The full static Einstein universe is of general physical interest as it is conformally equivalent to all closed FRW metrics. The Einstein metric is

$$ds^2 = -dt^2 + a^2(d\chi^2 + \sin^2\chi\, d\Omega^2), \tag{2}$$

where $\chi \in [0, \pi]$, $\theta \in [0, \pi]$, and $\phi \in [0, 2\pi]$. The scale factor a is related to the cosmological constant Λ through $a = \Lambda^{-1/2} = (4\pi G\rho_0)^{-1/2}$, where ρ_0 is the energy density of matter (dust). The vacuum part of the energy density is $\Lambda/8\pi G$; the pressure is $p = -\Lambda/8\pi G$.

The *half* Einstein universe is characterized by χ being restricted to one half of the previous interval, $\chi \in [0, \pi/2]$. A general point worth noticing in connection with this universe is that it is conformally related to anti-de Sitter space and is of relevance to supersymmetry [11]. The universe can be envisioned as a three-dimensional spherical volume spanned by the 'radius' χ and the angular coordinates θ and ϕ, closed by a two-dimensional spherical surface at $\chi = \pi/2$. We will take this surface to be perfectly conducting, this being an analogy to the Dirichlet boundary in the case of a scalar field.

As there are two dimensional parameters in the problem, viz. β and a, it becomes natural to introduce

$$\delta = \frac{\beta}{2\pi a} \tag{3}$$

as a nondimensional parameter. At high temperatures $\delta \ll 1$, and can thus be used as expansion parameter in a perturbative analysis. As anticipated above, it is the case of small δ's that is of thermodynamic interest. This can be seen from the following physical argument: In order to apply thermodynamic formalism to a fluctuating quasi-classical system, the temperature has to be sufficiently high to satisfy the condition $T \gg 1/\tau$, τ being the

relaxation time [16]. In our case we may take τ to be of the same order as the transit time for light across a distance a, *i.e.,* $\tau \sim a$. [This is essentially the same kind of situation as experienced when a narrow beam of light impinges upon a liquid surface and makes it bulge outward; cf. the classic light pressure experiment of Ashkin and Dziedzic [17] and also the theoretical discussion of it in Ref. [18]. In that case, the relaxation time is of the same order as the transit time for *sound* to traverse the beam.] Thus, in the present case $T \gg 1/a$, which means that

$$\delta \ll 1 \tag{4}$$

is the condition for applicability of thermodynamics.

It is of interest at this point to make a comparison with the Casimir effect. Let us assume the usual Casimir configuration, namely two parallel metal plates separated by a gap a. At finite temperature we are dealing with discrete Matsubara frequencies, $\zeta_m = 2\pi m T$, with $m \geq 0$ an integer. It turns out that the most important contributions to the Casimir force come from the frequency region $\zeta_m a \sim 1$, or $m \sim 1/(2\pi a T)$ (cf. the discussion on this point in [19]). That is, $m \sim \delta$. Then, we may interpret the parameter δ physically as the magnitude of the most important Matsubara numbers that occur in the analogous Casimir effect. This correspondence appears to be physically reasonable, though not trivial.

III. GOVERNING EQUATION. THE TM MODES

We take an orthonormal basis, $(ad\chi, a\sin\chi d\theta, a\sin\chi\sin\theta d\phi)$, and split off the time factor as $e^{-i\omega t}$. From Maxwell's equations in curvilinear space we can derive the governing equation for E_χ (or H_χ). Denoting collectively these field components by X, and writing the angular contributions as spherical harmonics, $X(\chi,\theta,\phi) = X(\chi)Y_{lm}(\theta,\phi)$, we obtain [1]

$$\frac{d^2}{d\chi^2}\left(\sin^2\chi\, X\right) + (\omega a)^2 \sin^2\chi\, X - l(l+1)X = 0. \tag{5}$$

The solution to this equation is

$$X \propto \sin^{l-1}\chi\, C_{n-l}^{(l+1)}(\cos\chi), \tag{6}$$

where n is an integer and $C_{n-l}^{(l+1)}$ are the Gegenbauer polynomials [20] satisfying the differential equation

$$(1-x^2){C_p^{(\alpha)}}''(x) - (2\alpha+1)x\, {C_p^{(\alpha)}}'(x) + p(p+2\alpha)C_p^{(\alpha)}(x) = 0 \tag{7}$$

for $p \geq 1$. Inserting X into Eq. (5) we obtain the eigenfrequencies

$$\omega_n = \frac{n+1}{a}, \quad n \geq l \geq 1. \tag{8}$$

To avoid infinities at the origin $\chi = 0$, we must have $n - l \geq 0$ [4].

So far, the boundary conditions have not been considered. The transverse magnetic modes are subject to the boundary condition

$$\partial_\chi(\sin\chi\,\mathbf{H}_\perp) = 0, \quad \chi = \frac{\pi}{2}, \tag{9}$$

where $\mathbf{H}_\perp$ is the magnetic field component transverse to the radius χ. From Maxwell's equations in orthonormal basis [4] we find, when comparing with the general solutions of the governing equation (5), that the magnitude $H_\perp$ of the vector $\mathbf{H}_\perp$ must be of the form

$$H_\perp \propto \sin^l\chi\, C_{n-l}^{(l+1)}(\cos\chi). \tag{10}$$

Thus we have from Eq. (9)

$$\partial_\chi \left\{ \sin^{l+1}\chi\, C_{n-l}^{(l+1)}(\cos\chi) \right\} = 0 \tag{11}$$

at the boundary. Observing the recursion relation for the derivatives of the Gegenbauer polynomials we get

$$\partial_x C_n^{(\alpha)}(0) = (n + 2\alpha - 1) C_{n-1}^{(\alpha)}(0) \tag{12}$$

at the boundary. The condition (11) now yields

$$C_{n-l-1}^{(l+1)} = 0. \tag{13}$$

As the Gegenbauer polynomials $C_m^{(\alpha)}$ vanish for odd m, $(n-l)$ in Eq. (13) must be even. It follows that for n even (odd), l must be even (odd) and the degeneracies are

$$g_n^{(e)} = \sum_{l=2,4,6,\ldots} (2l+1) = \frac{n-1}{2}(n+2), \quad n \quad \text{even}, \tag{14}$$

$$g_n^{(o)} = \sum_{l=1,3,5,\ldots} (2l+1) = \frac{n}{2}(n+1), \quad n \quad \text{odd}. \tag{15}$$

This leads to the following expression for the logarithm of the partition function Z^{TM} for the TM modes:

$$\ln Z^{\mathrm{TM}} = -\sum_{n=1}^{\infty}(2n-1)(n+1)\ln\left[1 - e^{-2\pi(2n+1)\delta}\right] - \sum_{n=1}^{\infty}(2n-1)n\ln\left[1 - e^{-4\pi n\delta}\right]. \tag{16}$$

From this we identify two types of sums:

$$G^{(\alpha)} = \sum_{n=1}^{\infty} n^{\alpha} \ln\left[1 - e^{-2\pi(2n+1)\delta}\right], \quad \alpha = 0, 1, 2, \tag{17}$$

and

$$H^{(\alpha)} = \sum_{n=1}^{\infty} (2n-1)^{\alpha} \ln\left[1 - e^{-4\pi n\delta}\right], \quad \alpha = 1, 2, \tag{18}$$

so that

$$\ln Z^{\mathrm{TM}} = -2G^{(2)} - G^{(1)} + G^{(0)} - \frac{1}{2}(H^{(2)} + H^{(1)}). \tag{19}$$

We expand the logarithm in Eq. (17) and take the derivative with respect to δ:

$$\begin{aligned} \frac{\partial G^{(\alpha)}}{\partial \delta} &= 2\pi \sum_{n=1}^{\infty} n^{\alpha}(2n+1) \sum_{k=1}^{\infty} e^{-2\pi(2n+1)k\delta} \\ &= \frac{1}{i} \int_C \mathrm{ds}(2\pi\delta)^{-\mathrm{s}}\, \Gamma(\mathrm{s})\zeta(\mathrm{s}) \sum_{\mathrm{n}=1}^{\infty} \mathrm{n}^{\alpha}(2\mathrm{n}+1)^{1-\mathrm{s}}, \end{aligned} \tag{20}$$

$\zeta(s)$ being Riemann's zeta function. In the last step above we made use of the relation

$$e^{-x} = \frac{1}{2\pi i} \int_C ds\, x^{-s}\, \Gamma(s), \tag{21}$$

where the integration contour is a line parallel to the imaginary axis at a sufficiently large value of $\Re s$.

As we need to work out the sum over n it will prove worthwhile to derive a general expression for the integral in Eq. (20). To this end we change the summation variable,

$$\sum_{n=1}^{\infty} n^{\alpha}(2n+1)^{1-s} = \sum_{k=3,5,\ldots}^{\infty} \left(\frac{k-1}{2}\right)^{\alpha} k^{1-s}. \tag{22}$$

The terms in brackets can be expressed as a binomial series. We add and subtract the $k=1$ term as well as the even terms, and insert the result into Eq. (20) to get

$$\begin{aligned} \frac{\partial G^{(\alpha)}}{\partial \delta} &= \frac{1}{2^{\alpha} i} \int_C \mathrm{ds}(2\pi\delta)^{-\mathrm{s}}\, \Gamma(\mathrm{s})\zeta(\mathrm{s}) \\ \times \sum_{l=0}^{\alpha} \begin{pmatrix} \alpha \\ l \end{pmatrix} &(-1)^{l} \Big\{(1 - 2^{1-s+\alpha-l})\zeta(s-1-\alpha+l) - 1\Big\}. \end{aligned} \tag{23}$$

When $\alpha = 0$ this expression becomes

$$\frac{\partial G^{(0)}}{\partial \delta} = \frac{1}{i} \int_C \mathrm{ds}(2\pi\delta)^{-\mathrm{s}}\Gamma(\mathrm{s})\zeta(\mathrm{s}) \left\{(1 - 2^{1-\mathrm{s}})\zeta(\mathrm{s}-1) - 1\right\}. \tag{24}$$

The two terms in the integrand have poles for $s = 0, 2$ and $s = 0, 1$ respectively. There are thus three poles in all.

When $\alpha = 1$ we have

$$\frac{\partial G^{(1)}}{\partial \delta} = \frac{1}{2i} \int_C \mathrm{ds} (2\pi\delta)^{-\mathrm{s}} \Gamma(\mathrm{s}) \zeta(\mathrm{s}) \left\{ (1 - 2^{2-\mathrm{s}}) \zeta(\mathrm{s} - 2) - (1 - 2^{1-\mathrm{s}}) \zeta(\mathrm{s} - 1) \right\}. \tag{25}$$

The poles are at $s = 1, 3$ in the first term, and at $s = 0, 2$ in the last term.

The final integral is for $\alpha = 2$,

$$\begin{aligned} \frac{\partial G^{(2)}}{\partial \delta} &= \frac{1}{4i} \int_C \mathrm{ds} (2\pi\delta)^{-\mathrm{s}} \, \Gamma(\mathrm{s}) \zeta(\mathrm{s}) \qquad (26) \\ &\times \left\{ (1 - 2^{3-s}) \zeta(s - 3) - 2(1 - 2^{2-s}) \zeta(s - 2) + (1 - 2^{1-s}) \zeta(s - 1) \right\}, \end{aligned}$$

with poles at $s = 0, 4$, $s = 1, 3$ and $s = 0, 2$ in the three terms respectively. Calculating all residues and collecting terms we find

$$\frac{\partial G^{(0)}}{\partial \delta} = \frac{\pi}{24\delta^2} - \frac{1}{\delta} + \frac{11\pi}{12}, \tag{27}$$

$$\frac{\partial G^{(1)}}{\partial \delta} = \frac{\zeta(3)}{8\pi^2\delta^3} - \frac{\pi}{48\delta^2} + \frac{1}{24\delta} + \frac{\pi}{24}, \tag{28}$$

$$\frac{\partial G^{(2)}}{\partial \delta} = \frac{\pi}{960\delta^4} - \frac{\zeta(3)}{8\pi^2\delta^3} + \frac{\pi}{96\delta^2} - \frac{1}{24\delta} - \frac{\pi}{160}. \tag{29}$$

We turn now to $H^{(\alpha)}$, following the same steps as for $G^{(\alpha)}$. First, we express the derivative in the form of an integral,

$$\frac{\partial H^{(\alpha)}}{\partial \delta} = \frac{2}{i} \int_C \mathrm{d}s (4\pi\delta)^{-s} \Gamma(s) \zeta(s) \sum_{n=1}^{\infty} n^{1-s} (2n - 1)^{\alpha}. \tag{30}$$

Again using the binomial series in the n sum we arrive at the following generic expression:

$$\begin{aligned} \frac{\partial H^{(\alpha)}}{\partial \delta} &= \frac{2}{i} \int_C \mathrm{ds} (4\pi\delta)^{-\mathrm{s}} \Gamma(\mathrm{s}) \zeta(\mathrm{s}) \\ \times \sum_{l=0}^{\alpha} \begin{pmatrix} \alpha \\ l \end{pmatrix} &(-1)^l 2^{\alpha - l} \zeta(s - 1 - \alpha + l). \qquad (31) \end{aligned}$$

For $\alpha = 1$ we here get

$$\frac{\partial H^{(1)}}{\partial \delta} = \frac{2}{i} \int_C \mathrm{ds} (4\pi\delta)^{-\mathrm{s}} \, \Gamma(\mathrm{s}) \zeta(\mathrm{s}) \left\{ 2\zeta(\mathrm{s} - 2) - \zeta(\mathrm{s} - 1) \right\}, \tag{32}$$

with poles at $s = 1, 3$ in the first term and $s = 0, 1, 2$ in the second term. Similarly, for $\alpha = 2$ we get

$$\frac{\partial H^{(2)}}{\partial \delta} = \frac{2}{i} \int_C \mathrm{ds} (4\pi\delta)^{-\mathrm{s}} \, \Gamma(\mathrm{s}) \zeta(\mathrm{s}) \left\{ 4\zeta(\mathrm{s} - 3) - 4\zeta(\mathrm{s} - 2) + \zeta(\mathrm{s} - 1) \right\}, \tag{33}$$

with poles at $s = 0, 4$, $s = 1, 3$ and $s = 0, 1, 2$ in the first, second and third term respectively. Calculating all residues we obtain

$$\frac{\partial H^{(1)}}{\partial \delta} = \frac{\zeta(3)}{4\pi^2\delta^3} - \frac{\pi}{24\delta^2} + \frac{1}{3\delta} - \frac{\pi}{6}, \tag{34}$$

$$\frac{\partial H^{(2)}}{\partial \delta} = \frac{\pi}{240\delta^4} - \frac{\zeta(3)}{2\pi^2\delta^3} + \frac{\pi}{24\delta^2} - \frac{1}{6\delta} + \frac{\pi}{10}. \tag{35}$$

Inserting the various terms into Eq. (19) we obtain the following expression for the energy $E^{TM} = -\partial/\partial\beta \ln Z^{TM}$ of the TM modes:

$$2\pi a E^{\mathrm{TM}} = \frac{\pi}{240\delta^4} - \frac{\zeta(3)}{4\pi^2\delta^3} - \frac{\pi}{24\delta^2} + \frac{25}{24\delta} - \frac{221\pi}{240}. \tag{36}$$

We similarly calculate the free energy $F^{TM} = -(1/\beta)\ln Z^{TM}$:

$$\beta F^{TM} = -\frac{\pi}{720\delta^3} + \frac{\zeta(3)}{8\pi^2\delta^2} + \frac{\pi}{24\delta} + \frac{25}{24}\ln\delta - \frac{221\pi}{240}\delta, \tag{37}$$

and finally the entropy $S^{TM} = \beta^2\partial F^{TM}/\partial\beta$:

$$S^{TM} = \frac{\pi}{180\delta^3} - \frac{3\zeta(3)}{8\pi^2\delta^2} - \frac{\pi}{12\delta} - \frac{25}{24}\ln\delta + \frac{25}{24}. \tag{38}$$

To leading order in δ this yields

$$\frac{S^{TM}}{E^{TM}} = \frac{4}{3}\beta, \quad \delta \ll 1. \tag{39}$$

Thus, in the high temperature limit the entropy of the TM modes is bounded, just as for the TE modes; cf. Eq. (1).

IV. COMPARISON WITH THE TE MODES. THE TOTAL FIELD QUANTITIES

The equality of the entropy/energy ratios for the TM and TE modes at high temperatures - meaning physically, as we have seen - that the temperature T is much higher than the inverse relaxation time $1/\tau$ - is as we might expect in view of the separability of the TM and TE modes in spherical geometry (cf., for instance, Ref. [21]). But according to our calculations there are differences between these modes as regards higher order terms in $\delta = \beta/2\pi a$. The expressions pertaining to the TE modes are

$$2\pi a E^{\mathrm{TE}} = \frac{\pi}{240\delta^4} - \frac{\zeta(3)}{4\pi^2\delta^3} - \frac{\pi}{24\delta^2} + \frac{13}{24\delta} - \frac{41\pi}{240}, \tag{40}$$

$$\beta F^{\mathrm{TE}} = -\frac{\pi}{720\delta^3} + \frac{\zeta(3)}{8\pi^2\delta^2} + \frac{\pi}{24\delta} + \frac{13}{24}\ln\delta - \frac{41\pi}{240}\delta, \tag{41}$$

$$S^{\mathrm{TE}} = \frac{\pi}{180\delta^3} - \frac{3\zeta(3)}{8\pi^2\delta^2} - \frac{\pi}{12\delta} - \frac{13}{24}\ln\delta + \frac{13}{24}. \tag{42}$$

These results were obtained in [1] via the same method as above, and also, as an independent check, via use of the Euler-Maclaurin sum formula. Adding the contributions from the TM and TE modes we obtain the following total field quantities, when expressed in conventional units,

$$E = \frac{\pi^4}{15}a^3T^4 - 2\zeta(3)a^2T^3 - \frac{\pi^2}{6}aT^2 + \frac{19}{12}T - \frac{131}{240a}, \tag{43}$$

$$F = -\frac{\pi^4}{45}a^3T^4 + \zeta(3)a^2T^3 + \frac{\pi^2}{6}aT^2 - \frac{19}{12}T\ln(2\pi aT) - \frac{131}{240\pi a}, \tag{44}$$

$$S = \frac{4\pi^4}{45}a^3T^3 - 3\zeta(3)a^2T^2 - \frac{\pi^2}{3}aT + \frac{19}{12}\ln(2\pi aT) + \frac{19}{12}. \tag{45}$$

Surprisingly enough, the TE and TM contributions to the T^3 and T^2 terms in the expression (43) for E do not cancel out.

SUMMARY AND FURTHER DISCUSSION

Let us first summarize a couple of points:

• We have assumed the parameter $\delta = \beta/2\pi a = 1/(2\pi aT)$ to be small. This is in accordance with the requirement of classical thermodynamics: under equilibrium conditions T has to be much larger than the inverse relaxation time [16]. When comparing with the Casimir effect between two metal plates, δ may be given a physical interpretation as the magnitude of the most dominant Matsubara numbers [19].

• The most striking result of the above calculation is that the TE and TM modes do not compensate each other to orders T^3 and T^2 in the expression (43) for the total energy. One might expect beforehand that the mentioned compensation should take place here as well as in the known case of flat space, considered earlier in connection with the Casimir effect [8, 9]. We attribute the non-compensation to the properties of the Einstein metric. The formalism is generally too complicated to be transparent beforehand (it may be noted here that the degeneracies of the TM modes as given in Eqs. (14) and (15) are complementary to those holding for the TE modes [1]).

• We shall consider below some aspects related to dispersion. As a preliminary step, let us give first a brief account of the essence of the formalism for calculating the total energy

associated with the individual TE and TM field oscillation modes, characterized by the numbers n and l [4].

Consider first the TE modes. As in [4] it is convenient to change the meaning of n, such that n runs from 0 upwards. The eigenfrequencies can then be expressed as

$$\omega_n^{TE} = \frac{2n+l+2}{a}, \tag{46}$$

with $l = 1, 2, 3...$ as before. The "radial" magnetic field component can be written as

$$H_\chi = A^{TE} l(l+1) \sin^{l-1}\chi \, C_{2n+1}^{(l+1)}(\cos\chi) \, Y_{lm}(\theta,\phi), \tag{47}$$

where A^{TE} is a normalization constant. The other magnetic field components are, in an orthonormal basis,

$$H_\theta = \frac{A^{TE}}{\sin\chi} \frac{d}{d\chi} \left[\sin^{l+1}\chi \, C_{2n+1}^{(l+1)}(\cos\chi) \right] \partial_\theta Y_{lm}, \tag{48}$$

$$H_\phi = \frac{imA^{TE}}{\sin\chi} \frac{d}{d\chi} \left[\sin^{l+1}\chi \, C_{2n+1}^{(l+1)}(\cos\chi) \right] \frac{Y_{lm}}{\sin\theta}. \tag{49}$$

The time factor $\exp(-i\omega t)$ is assumed everywhere. The electric field components E_θ and E_ϕ, not given here, follow from Maxwell's equations. Using these field expressions, we obtain by integrating over the volume the following expression for the total energy:

$$E_{nl}^{TE} = \frac{\pi}{8} a^3 |A^{TE}|^2 \, l(l+1)(2n+l+2) \, \frac{2^{-2l}\Gamma(2n+2l+3)}{(2n+1)! \, [\Gamma(l+1)]^2}. \tag{50}$$

As for the TM modes, we write analogously

$$\omega_n^{TM} = \frac{2n+l+1}{a}, \tag{51}$$

with $n = 0, 1, 2, ...$ The electric field components can be written as

$$E_\chi = A^{TM} l(l+1) \sin^{l-1}\chi \, C_{2n}^{(l+1)}(\cos\chi) \, Y_{lm}, \tag{52}$$

$$E_\theta = \frac{A^{TM}}{\sin\chi} \frac{d}{d\chi} \left[\sin^{l+1}\chi \, C_{2n}^{(l+1)}(\cos\chi) \right] \partial_\theta Y_{lm}, \tag{53}$$

$$E_\phi = \frac{imA^{TM}}{\sin\chi} \frac{d}{d\chi} \left[\sin^{l+1}\chi \, C_{2n}^{(l+1)}(\cos\chi) \right] \frac{Y_{lm}}{\sin\theta}, \tag{54}$$

with corresponding expressions for the transverse components H_θ and H_ϕ. The total energy becomes in this case

$$E_{nl}^{TM} = \frac{\pi}{8} a^3 |A^{TM}|^2 l(l+1)(2n+l+1) \, \frac{2^{-2l} \, \Gamma(2n+2l+2)}{(2n)! \, [\Gamma(l+1)]^2}. \tag{55}$$

• We are now able to discuss how frequency dispersion restricts the modes of oscillations in the cavity. As for dispersion relation, we may for a dielectric take a Lorentz (or Sellmeir) form as mentioned already in the Introduction,

$$\varepsilon(i\zeta) = 1 + \frac{\varepsilon(0) - 1}{1 + \zeta^2/\omega_0^2}, \tag{56}$$

where ζ is the imaginary frequency and ω_0 the resonance frequency, the latter being a soft frequency cutoff. However, there are no thermodynamic restrictions preventing us from assuming that there is a simple sharp cutoff at $\zeta = \omega_0$, so let us adopt this simple prescription. Moreover, when assuming an ideal metal, we have that $\varepsilon(0) = \infty$. Our dispersion model becomes accordingly

$$\varepsilon(i\zeta) = \begin{cases} \infty, & \zeta \leq \omega_0 \\ 1, & \zeta > \omega_0. \end{cases} \tag{57}$$

As for resonance frequency ω_0, we shall take the same typical value as mentioned earlier,

$$\omega_0 = 1.5 \times 10^{16}\,\text{rad/s}. \tag{58}$$

Consider now the TE modes, where the eigenfrequencies are given in dimensional units as $\omega_n^{TE} = (c/a)(2n + l + 2)$. The lowest mode is obtained for $n = 0, l = 1$ as $(\omega_0^{TE})_{min} = 3c/a$. Let us choose the "radius" of the cavity to be small,

$$a = 45\,\text{nm} \tag{59}$$

(this radius is large enough to permit use of macroscopic electromagnetic theory in the material). Then, $(\omega_0^{TE})_{min} = 2 \times 10^{16}$ rad/s. This mode can according to (58) *not* exist in the cavity; the permittivity in the walls is simply equal to one.

In the TM case, we obtain analogously from $\omega_n^{TM} = (c/a)(2n+l+1)$ that $(\omega_0^{TM})_{min} = 2c/a$ for $n = 0, l = 1$. This is the lowest possible oscillation mode in the cavity (if we assume that there is no restriction coming from dispersion at all). Numerically, $(\omega_0^{TM})_{min} = 1.33 \times 10^{16}$ rad/s. This oscillation mode can thus exist also in the present case, under the given conditions. Our choice of values in Eqs. (58) and (59) has thus managed to make the lowest possible mode in the cavity to be the only real one. Our example is somewhat extreme, but it serves to demonstrate the important effects of dispersion.

Equations (50) and (55) permit us to calculate the field energy in each mode. In the present case only (55) is actual, and it gives for the lowest mode

$$E_{01}^{TM} = \frac{3\pi}{4} a^3 |A^{TM}|^2. \tag{60}$$

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